**Quantized Elastic Field**

Now let’s do it in 3D. We’ll recall from the Classical Mechanics folder, we have, in the simplest (i.e., isotropic) approximation:



where the **:** is a double dot product (between two tensors). Explicitly, this is:



(implicit summation over repeated indices) So our Lagrangian is:



We can get the equation of motion of our field by minimizing the action.



and,



Now integrate by parts to get the derivatives off of the δ’s,



Should recognize the entity operating on φα, on the right, as the Laplacian. So we have, dividing by ρ,



And for an elastic solid we’d also impose boundary conditions on each component:



[assuming periodic boundary conditions, say]. To solve, we’d expand φ in terms of the eigenfunctions of the spatial part.



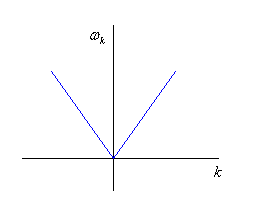
So we have to find its eigenfunctions.

They are clearly



where v is the velocity of waves in the solid. This is the conventional dispersion relation for harmonic waves in a solid, in the long wavelength limit [recall going to continuum screws up the real dispersion relation].



which is just the low wavelength limit of the discrete case. Plugging this into our equation of motion, it will reduce to



and the solution to this equation is done via usual ODE stuff. We get:



So that the general solution of the equation of motion is



We’re going to want some Q properties and commutation relations in just a second. So let’s work on those. First off, since φ is Hermitian, we must have:



and so we can identify,



And now we can work out the Q commutation relations by demanding they reproduce the φ commutation relations. I’ll do so at t = 0 for convenience,



Can see this necessitates,



Now we’re going to construct the ‘free-field expansion’ of φ. Recall from QM that we can identify operators which develop harmonically as annihilators/creators of energy excitations. So looking back at φ(x,t) we know now that the energy excitations are just ωk. And up to normalization, the annihilation/creation operators are:



We can determine the normalization constant by imposing:



So we have:



[noting ωk is symmetric w/r to k] And so now we can write:



The vector field can be written as, introducing the Cartesian basis vectors α.



where M = ρV is the total mass. But actually we usually write it another way – probably because interactions with other fields most often couple to the elastic field through its excitations’ momentum vectors. So we use a **k**-dependent basis, **k**λ, which points along and perpendicular to **k**.



Waves going along **εk**3 are said to be longitudinally polarized, and those along **εk**1,2 or some linear combination of them would be transversely polarized. Interactions generally couple to lattice oscillations in the longitudinal direction, so that is why we take the trouble to write the oscillations in this basis. We can put our field in terms of this basis by dotting it against the unit tensor comprised of this new basis:



So,



where **k**λ(α) is the component of **k**λ in the αth direction. Now the creation/annihilation operator in the λ direction as:



We can verify that these satisfy the usual canonical commutation relations:



This equality can be obtained from the relationship:



So…we can write our FFE as:



(remember M = ρV, so, total mass) We could go on and put this FFE into H and see what we get. So,



Continuing,



So we get:



Good. If we had used the Cartesian basis, I think we’d have gotten:



We can write down the wavefunctional again. Generalizing from the 1D case, just as we did in the QM/Many Particles/Distinct/N harmonic oscillators file, I think we can say:



where we recall *a* is the lattice spacing, and μ = ρL2 is the linear mass density. So this is our wavefunctional.

**Checking some things…**

Do φ and π have proper commutation relations?



Great.